

On the interaction between two point electric charges

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Abstract

The general formula for the interaction potential between two point electric charges is derived. This analytical formula has the correct asymptotic behaviour at large distances between two interacting charges. The derivation of this formula is based on the closed analytical expression for the Uehling potential obtained earlier (A.M. Frolov and D.M. Wardlaw, ArXiv: 1110.3433 [nucl-th]).

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In classical electrodynamics the interaction V between two point electric charges q_1e and q_2e is described by the Coulomb law, i.e. $V \simeq \frac{q_1q_2e^2}{r_{12}}$. This expression for V does not change its form in the non-relativistic quantum mechanics [1], when the Planck constant \hbar is finite. However, if we also assume that the speed of light c is finite, then Quantum Electrodynamics leads us to the necessity to modify the Coulomb law. The main correction is related to the vacuum polarization in the spatial areas close to the electric charges. In the lowest order approximation such a correction is represented by the Uehling potential [2], which is correct at short interparticle distances. However, the Uehling potential does not provide the correct long-range asymptotics, i.e. its behaviour at large r is not correct. Therefore, it must be modified to represent the interaction between the two point electric charges correctly at arbitrary, in principle, interparticle distances. This problem has substantial scientific and methodological interest. In particular, the Coulomb potential modified by a number of additional terms, which represent small QED-corrections, can be used directly in highly accurate calculations of bound states in few-electron atoms and ions. On the other hand, it is interesting to see the transformation of the usual Coulomb potential under the rules of modern Quantum Electrodynamics.

In our earlier study [3] we have derived the closed analytical expression for the Uehling potential $U(r)$ which describes the lowest order QED correction (upon the fine structure constant α) to the regular Coulomb potential. This correction is related with the polarization of the vacuum produced by a point electric charge. In [3] it was shown that the total interaction potential is written in the form (in atomic units $\hbar = 1, m_e = 1$ and $e = 1$)

$$\begin{aligned} \phi(r) = \frac{Q}{r} \left\{ 1 + \frac{2\alpha}{3\pi} \cdot \left[\left(1 + \frac{r^2}{3\alpha^2} \right) K_0\left(\frac{2r}{\alpha}\right) - \frac{r}{6\alpha} K_{i_1}\left(\frac{2r}{\alpha}\right) - \left(\frac{5}{6} + \frac{r^2}{3\alpha^2} \right) K_{i_2}\left(\frac{2r}{\alpha}\right) \right] \right\} \\ = \frac{Q}{r} + U(r) \end{aligned} \quad (1)$$

where $U(r)$ is the Uehling potential [2] and $\alpha = \frac{e^2}{\hbar c} = c^{-1}$ is the fine structure constant. Here and everywhere below in this study the notation \hbar stands for the reduced Planck constant ($\hbar = \frac{h}{2\pi}$), e designates the absolute value of the electric charge of the electron and c is the speed of light in vacuum. In Eq.(1) the function $K_0(a)$ is the modified Bessel function of zero order (see Eqs.(8.432) and (8.447) from [4]), i.e.

$$K_0(z) = \int_0^\infty \exp(-z \cosh t) dt = \sum_{k=0}^{\infty} (\psi(k+1) + \ln 2 - \ln z) \frac{z^{2k}}{2^{2k} (k!)^2} ,$$

where $\psi(k)$ is the Euler *psi*-function defined by Eq.(8.362) from [4]. The functions $K_{i_1}(z)$

and $Ki_2(z)$ in Eq.(1) are the recursive integrals of the $K_0(z)$ function, i.e.

$$Ki_1(z) = \int_z^\infty Ki_0(z)dz \quad , \quad \text{and} \quad Ki_n(z) = \int_z^\infty Ki_{n-1}(z)dz \quad , \quad (2)$$

where $n \geq 1$ and $Ki_0(z) \equiv K_0(z)$.

The formula, Eq.(1), is corret mathematically, but it has a wrong asymptotic at large distances r , e.g., at $r \geq 10\alpha a_0$, where $a_0 = \frac{\hbar^2}{m_e e^2}$ is the Bohr radius. First, note that the short range asymptotic of the $\phi(r)$ potential, Eq.(1), takes the form

$$\phi(r) |_{r \rightarrow 0} \simeq \frac{Q}{r} \left\{ 1 + \frac{\alpha}{3\pi} \left[-\frac{5}{3} - 2\gamma + 2 \ln \alpha - 2 \ln r \right] \right\} \quad (3)$$

where $\gamma \approx 0.5772156649 \dots$ is the Euler constant (see, e.g., [4]) and $\ln \alpha \approx -4.92024365857$. The long-range asymptotics of the potential $\phi(r)$, Eq.(1), is (in atomic units)

$$\phi(r) |_{r \rightarrow \infty} \simeq \frac{Q}{r} \left\{ 1 + \frac{\alpha^{\frac{5}{2}}}{4\sqrt{\pi} r^{\frac{3}{2}}} \exp\left(-\frac{2}{\alpha} r\right) \right\} \quad (4)$$

This means that the long-range asymptotics of $\phi(r)$ decreases with r exponentially. This has no physical sense, since the exponential function vanishes at $r \rightarrow +\infty$ very rapidly. The corresponding correction to the potential at large distances will always be outweighed by the lowest order QED correction to the electromagnetic field (or EM-field, for short) (\mathbf{E}, \mathbf{H}), or to the pure electric field \mathbf{E} in our case, when $\mathbf{H} = 0$. This correction provides power-type dependence at large distances and it is directly related to the non-linearity of the Maxwell equations for the EM-field [5], [6].

Let us obtain the lowest order QED correction to the electric field \mathbf{E} by considering one point electric charge e which is at rest in the center of coordinates $x = 0, y = 0$ and $z = 0$. The corresponding Lagrangian L which generates the lowest order QED correction to the energy of the electric field is written in the form (in regular units) (see, e.g., [5])

$$L = \frac{1}{2} \mathbf{E}^2 + \frac{e^4 \hbar}{360 \pi^2 m_e^4 c^7} \mathbf{E}^4 = \frac{1}{2} (\nabla \phi)^2 + \frac{e^4 \hbar}{360 \pi^2 m_e^4 c^7} (\nabla \phi)^4 = L[\phi(r)] \quad (5)$$

where $\mathbf{E} = -\nabla \phi(r)$. By varying the potential $\phi(r)$ in the following equation (the fundamental equation of least action [6])

$$\delta \int L[\phi(r)] r^2 dr = 0 \quad (6)$$

one finds the differential equation

$$\frac{d\phi}{dr} + \frac{e^4 \hbar}{90 \pi^2 m_e^4 c^7} \left(\frac{d\phi}{dr} \right)^3 = \frac{C}{r^2} \quad (7)$$

where $C = -\frac{Qe}{4\pi}$. Assuming that $\phi(r) = \frac{Qe}{4\pi r} + \psi(r)$, where $\psi(r)$ is a very small correction we reduce the last equation to the form

$$\frac{d\psi}{dr} = \frac{e^4\hbar}{90\pi^2 m_e^4 c^7} \left(\frac{Q^3 e^3}{64\pi^3 r^6} \right) \quad (8)$$

From this equation one finds

$$\psi(r) = -\frac{e^7\hbar}{450\pi^2 m_e^4 c^7} \left(\frac{Q^3}{64\pi^3 r^5} \right) \quad (9)$$

Therefore, the total interaction potential $\phi(r)$ is

$$\phi(r) = \frac{Qe}{4\pi r} - \frac{e^7\hbar}{450\pi^2 m_e^4 c^7} \left(\frac{Q^3}{64\pi^3 r^5} \right) = \frac{Qe}{4\pi r} \left[1 - \frac{2\hbar}{225\pi m_e^4 c^7} \left(\frac{Q^2 e^6}{64\pi^3} \right) \frac{1}{r^4} \right] \quad (10)$$

In the relativistic units $\hbar = 1, c = 1$ and $\alpha = \frac{e^2}{4\pi}$ the last formula takes the form

$$\phi(r) = \frac{Qe}{4\pi r} \left[1 - \frac{2Q^2\alpha^3}{225\pi m_e^4 r^4} \right] \quad (11)$$

This expression is the sum of the Coulomb potential and small correction which exactly coincides with the Wichmann-Kroll potential [6]. In atomic units the expression, Eq.(11), takes the form

$$\phi(r) = \frac{Q}{r} \left[1 - \frac{2Q^2\alpha^7}{225\pi r^4} \right] = \frac{Q}{r} - \frac{2Q^3\alpha^7}{225\pi r^5} \quad (12)$$

and Wichmann-Kroll potential $W_K(r)$ is

$$\psi(r) = W_K(r) = -\frac{2Q^3\alpha^7}{225\pi r^5} \quad (13)$$

Note that this potential is not regular at the origin. However, as follows from the Appendix such a singularity is formal, since Eq.(7) is not correct at very small and very large r (see Appendix). The singularity at small r can be removed by using substitution $r \rightarrow r + \alpha$. Such a substitution removes singularity of the $W_K(r)$ potential at $r = 0$ and changes its behaviour at short interparticle distances $r \leq \alpha$. However, at such distances the Wichmann-Kroll correction is not important and its contribution is significantly smaller than contribution from the Uehling potential. On the other hand, at large distances, i.e. for $r \geq 10\alpha a_0$ (and even for $r \geq \alpha a_0$) the contribution from the Wichmann-Kroll potential is substantial and cannot be neglected (the correction from Uehling potential vanishes exponentially with r). Finally, we can write the correct expression for the Wichmann-Kroll potential $W_K(r)$

$$\psi(r) = W_K(r) = -\frac{2Q^3\alpha^7}{225\pi(r + \alpha)^5} \quad (14)$$

Note that this potential is always negative. In general, at $r \approx a_0$ this potential is very small by its absolute value. However, its overall contribution rapidly increases with the nuclear charge Q . Furthermore, it decreases with the distance r as $\sim r^{-5}$, i.e. non-exponentially. It is clear that at large distances the Wichmann-Kroll correction will always overweight the contribution from the Uehling potential $U(r)$, Eq.(1). Thus, the following interaction potential (in atomic units) of the two point electric charges

$$\Phi(r) = \frac{Q}{r} + U(r) + W_K(r) = \frac{Q}{r} + \frac{2Q\alpha}{3\pi r} \cdot \left[\left(1 + \frac{r^2}{3\alpha^2}\right) K_0\left(\frac{2r}{\alpha}\right) - \frac{r}{6\alpha} K_1\left(\frac{2r}{\alpha}\right) - \left(\frac{5}{6} + \frac{r^2}{3\alpha^2}\right) K_2\left(\frac{2r}{\alpha}\right) \right] - \frac{2Q^3\alpha^7}{225\pi(r+\alpha)^5} \quad (15)$$

has the correct asymptotic behaviour both at small and large interparticle distances r . In the case of interaction between two point electric charges q_1e and q_2e we need to replace in Eq.(15) the factor Q by the product q_1q_2 . The potential $\Phi(r_{12})$ takes the form (in atomic units)

$$\Phi(r_{12}) = \frac{q_1q_2}{r_{12}} + \frac{2q_1q_2\alpha}{3\pi r_{12}} \cdot \left[\left(1 + \frac{r_{12}^2}{3\alpha^2}\right) K_0\left(\frac{2r_{12}}{\alpha}\right) - \frac{r_{12}}{6\alpha} K_1\left(\frac{2r_{12}}{\alpha}\right) - \left(\frac{5}{6} + \frac{r_{12}^2}{3\alpha^2}\right) K_2\left(\frac{2r_{12}}{\alpha}\right) \right] - \frac{2(q_1q_2)^3\alpha^7}{225\pi(r_{12}+\alpha)^5} \quad (16)$$

where r_{12} is the distance between particles 1 and 2. In regular units one needs to replace in Eqs.(15) - (16) the dimensionless value α of the fine structure constant by the factor $\alpha \frac{\hbar^2}{m_e e^2}$. This produces the regular expression for the Wichmann-Kroll potential for atomic systems. Regularization of this term for systems of heavy particles requires an additional investigation.

Thus, we have derived the formula for the interaction between two point electric charges which includes the lowest order QED corrections to the Coulomb potential. Our formulas, Eqs.(15) - (16), provide the correct asymptotic behaviour at arbitrary interparticle distances. The formulas, Eqs.(15) - (16), can directly be used in highly accurate computations of the bound states in few-electron atoms and ions. The explicit computation of the electric field strength \mathbf{E} , i.e. the spatial gradient from the $\Phi(r_{12})$ potential, Eq.(16), is straightforward, but the Fourier resolution of the corresponding ‘electrostatic’ field is extremely complicated. Note also that for the potential $\Phi(r)$, Eq.(15), the condition $\Delta\Phi(r) = 4\pi Q\delta(\mathbf{r})$, where $\Delta = \nabla \cdot \nabla = \text{div}(\text{grad}...)$, is not obeyed.

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Appendix

The equation, Eq.(7), is a cubic equation in respect to the radial derivative of the unknown potential $\phi(r)$. It can be re-written in the form

$$\frac{e^4 \hbar}{90\pi^2 m_e^4 c^7} y^3 + y + \frac{Qe}{4\pi r^2} = 0 \quad (17)$$

where $y = \frac{d\phi}{dr}$. As follows from the general theory of cubic functions this equation has only one real root (the discriminant of this equation is negative). In fact, this equation is easily reduced to the form of a monic trinomial (see, e.g., [7])

$$y^3 + py + \frac{q}{r^2} = 0 \quad (18)$$

where $p = \frac{90\pi^2 m_e^4 c^7}{e^4 \hbar}$ and $q = \frac{45\pi Q m_e^4 c^7}{2e^3 \hbar}$.

The Cardano method gives the only real root of Eq.(18)

$$y = \sqrt[3]{-\frac{q}{2r^2} + \sqrt{\frac{q^2}{4r^4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2r^2} - \sqrt{\frac{q^2}{4r^4} + \frac{p^3}{27}}} \quad (19)$$

In other words, we obtain the following differential equation

$$\frac{d\phi}{dr} = \sqrt[3]{-\frac{q}{2r^2} + \sqrt{\frac{q^2}{4r^4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2r^2} - \sqrt{\frac{q^2}{4r^4} + \frac{p^3}{27}}} \quad (20)$$

At $r \rightarrow \infty$ this equation has the solution $\phi(r) = 2(\frac{p}{3})^{\frac{1}{2}} \cdot r + c$, where c is some numerical constant. At $r \rightarrow 0$ the analogous solution takes the form $\phi(r) = -3q^{\frac{1}{3}} \cdot r^{\frac{1}{3}} + c$, where c is some numerical constant. Both these asymptotics do not have any physical sense. Therefore, our procedure used in the main text to approximate the exact solution of Eq.(7) is correct only for some restricted interval of r -variation. In other words, the substitution $r \rightarrow r + \alpha$ used in the main text to regularize the Wichmann-Kroll potential at small r is correct.

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